CF

2\. (a)

$$\int 12x e^{x^2}(e^{x^2}+1)^2\,dx$$

let $u = e^{x^2}+1$

$\Rightarrow du = 2x e^{x^2}\,dx$

$\Rightarrow 6\,du = 12x e^{x^2}\,dx$

$$\int 6u^2\,du$$

$= 2u^3 + C$

$= 2\left[e^{x^2}+1\right]^3 + C$

(b)

$$\int_{0.2}^{1} \frac{9}{x^2}\sqrt{1+\frac{3}{x}}\,dx$$

$u = 1 + \frac{3}{x}$

$du = -\frac{3}{x^2}\,dx$

$\Rightarrow -3\,du = \frac{9}{x^2}\,dx$

$x = 0.2 \Rightarrow u = 16$

$x = 1 \Rightarrow u = 4$

$$\int_{16}^{4} -3u^{1/2}\,du$$

$= -2u^{3/2}\Big|_{16}^{4}$

$= -2(8) + 2(64)$

$= 112$

3 $\underset{\sim}{a} = \hat{i} + 2\hat{j} - 3k$

$\underset{\sim}{b} = \hat{i} + \hat{j} - \hat{k}$

let $\underset{\sim}{n} = x\hat{i} + y\hat{j} + z\hat{k}$ be normal vector

$\underset{\sim}{a} \cdot \underset{\sim}{n} = 0$ and $\underset{\sim}{b} \cdot \underset{\sim}{n} = 0$

$\Rightarrow$ $x + 2y - 3z = 0$ and $x + y - z = 0$

let $z = 1$

$\Rightarrow$
$$x + 2y = 3$$
$$x + y = 1$$
$$y = 2$$
$$x = -1$$

$$\underset{\sim}{n} = -\hat{i} + 2\hat{j} + \hat{k}$$

$$|\underset{\sim}{n}| = \sqrt{1+4+1}$$
$$= \pm\sqrt{6}$$

$$\hat{n} = \frac{1}{\pm\sqrt{6}}\left(-\hat{i} + 2\hat{j} + \hat{k}\right)$$

4. $$y = \log_{16} x^2$$

$$\Rightarrow \quad 16^y = x^2$$

$$\Rightarrow \quad y\ln 16 = 2\ln x$$

$$\Rightarrow \quad y = \frac{2\ln x}{\ln 16}$$

$$\frac{dy}{dx} = \frac{2}{x\ln 16}$$

$$y = \cos^3(e^{1-x^2})$$

$$\frac{dy}{dx} = 3\cos^2(e^{1-x^2})(-\sin(e^{1-x^2}))(-2x e^{1-x^2})$$

$$= 6x e^{1-x^2} \cdot \sin(e^{1-x^2}) \cos^3(e^{1-x^2})$$

$$y = \sqrt{x+y}$$

$$\Rightarrow \quad y^2 = x+y$$

$$\Rightarrow 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y-1}$$

when $y = 6$

$$\frac{dy}{dx} = \frac{1}{11}$$

when $y = 6$, $x = 30$

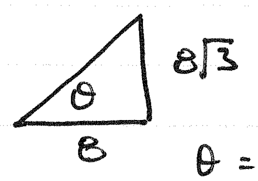
tangent

$$y = \frac{1}{11}x + c$$

$$\Rightarrow \quad y = \frac{1}{11}x + \frac{36}{11}$$

CF

5 $z^4 = 8 + 8\sqrt{3}i$



$\theta = \frac{\pi}{3}$

$\sqrt{8^2 + 8^2(3)} = 16$

$$z^4 = 16\,\text{cis}\left(\frac{\pi}{3} + 2k\pi\right)$$

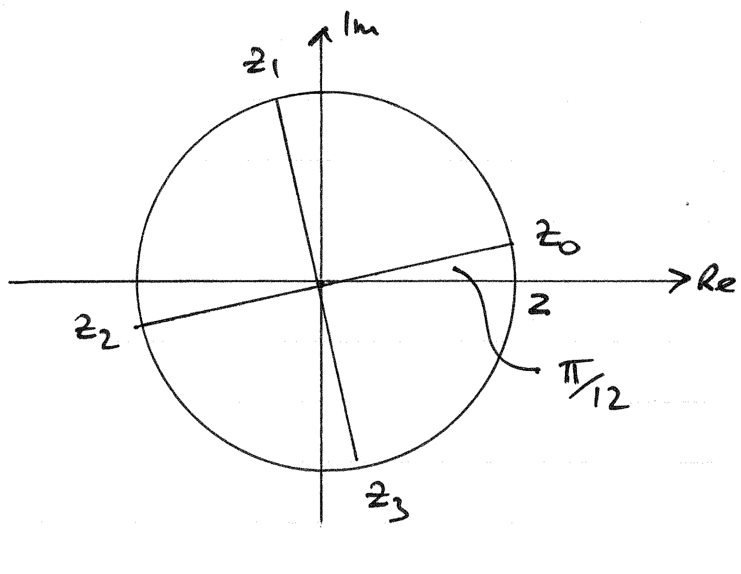
$$= 16\,\text{cis}\,\frac{(6k+1)\pi}{3} \qquad k = 0, 1, 2, 3$$

$k=0 \quad z = 2\,\text{cis}\,\frac{\pi}{12}$

$k=1 \quad z = 2\,\text{cis}\,\frac{7\pi}{12}$

$k=2 \quad z = 2\,\text{cis}\,\frac{13\pi}{12}$

$k=3 \quad z = 2\,\text{cis}\,\frac{19\pi}{12}$



CF

6 $A = \begin{bmatrix} a & 3 \\ 1 & a+2 \end{bmatrix}$

det A = $a(a+2) - 3$

singular iff det $(A) = 0$

$$a^2 + 2a - 3 = 0$$

$$(a+3)(a-1) = 0$$

$$\Rightarrow \quad a = -3 \text{ or } 1$$

$$P = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & -1 \\ 3 & -2 & 3 \end{bmatrix} \qquad Q = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad R = \begin{bmatrix} 9 \\ 13 \\ -9 \end{bmatrix}$$

$$AB + 2B = 4I$$
$$\Rightarrow (A + 2I)B = 4I$$
$$\Rightarrow (A + 2I) = 4B^{-1}$$
$$A = 4B^{-1} - 2I$$

or

$$AB + 2B = 4I$$
$$\Rightarrow AB = 4I - 2B$$
$$\Rightarrow A = (4I - 2B)B^{-1}$$

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

CF

7 to show $\dfrac{1+e^{a}}{1+e^{-a}} = e^{a}$

LHS $= \dfrac{1+e^{a}}{1+e^{-a}} \quad \dfrac{1-e^{-a}}{1-e^{-a}}$

$= \dfrac{1-e^{-a}+e^{a}-1}{1-e^{-2a}}$

$= \dfrac{e^{a}-e^{-a}}{1-e^{-2a}}$

$= \dfrac{e^{a}\left[1-e^{-2a}\right]}{1-e^{-2a}}$

$= e^{a}$

$=$ RHS

or LHS $= \dfrac{1+e^{a}}{1+e^{-a}} \quad \dfrac{e^{a}}{e^{a}}$

$= \dfrac{e^{a}(1+e^{a})}{e^{a}+1}$

$= e^{a}$

$=$ RHS

or LHS $= \dfrac{1+e^{a}}{1+e^{-a}}$

$= \dfrac{1+e^{a}}{1+\frac{1}{e^{a}}}$

$= e^{a}\left(\dfrac{1+e^{a}}{e^{a}+1}\right)$

$=$ RHS

CF

7

(b) $$\int_{-3}^{3} \frac{e^{kx}}{1+e^{kx}}\,dx \qquad u = 1+e^{kx}, \quad du = k e^{kx}\,dx$$

$$= \int \frac{1}{k}\frac{1}{u}\,du$$

$$= \frac{1}{k}\ln u + C$$

$$= \frac{1}{k}\ln\left(1+e^{kx}\right)\Big|_{-3}^{3}$$

$$= \frac{1}{k}\left[\ln\left(\cancel{e}1+e^{3k}\right) - \ln\left(1+e^{-3k}\right)\right]$$

$$= \frac{1}{k}\ln\left(\frac{1+e^{3k}}{1+e^{-3k}}\right)$$

$$= \frac{1}{k}\ln e^{3k}$$

$$= 3.$$

CA

8 $\frac{dm}{dt} = km$

$\Rightarrow \quad m = m_0 e^{kt}$

suburb A

$2 = e^{9.5k}$

$\Rightarrow \quad k = 0.0729$

$m_A = 55\,300\, e^{0.0729t}$

in 1989, $t = 5$

$m_A = 79\,646 \qquad \Rightarrow \quad m_A = 79\,646\, e^{0.0729t}$

suburb B

$2 = e^{8.5k}$

$\Rightarrow \quad k = 0.0815$

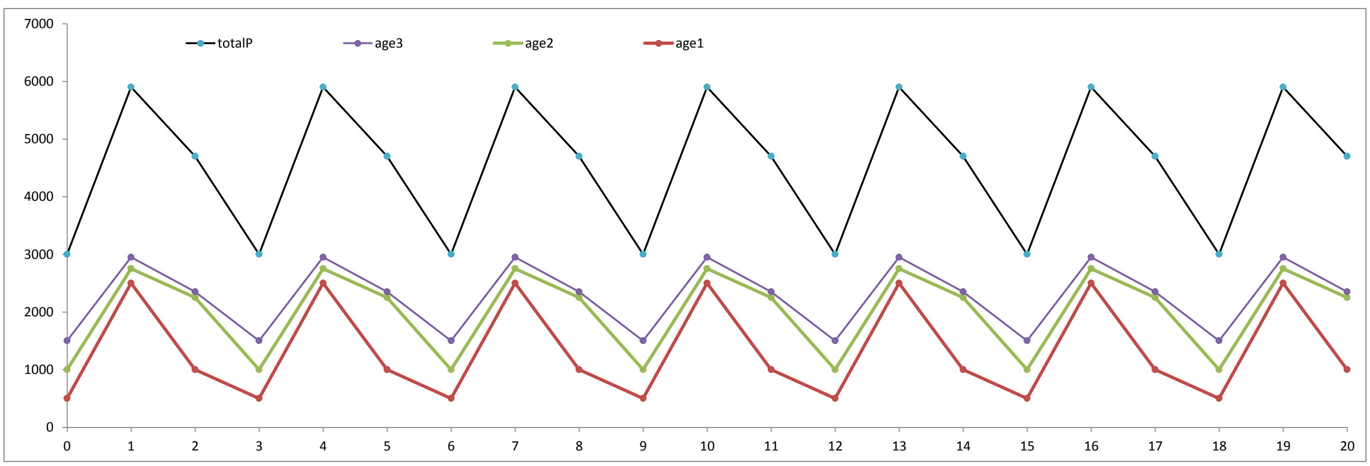
$m_B = 74100\, e^{0.0815t}$

$74100\, e^{0.0815t} = 79\,646\, e^{0.0729t}$

$\Rightarrow \quad t = 8.41$

prices were the same in 1989 + 8.4

1997



7000
6000
5000
4000
3000
2000
1000
0
totalP
age3
age2
age1
0
1
2
3
4
5
6
7
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9
10
11
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15
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17
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19
20

11 $f(x) = |x+2| + |3-2x|$

$x = -2 \qquad x = +3/2$

$x < -2 \qquad -2 \leq x < 3/2 \qquad x \geqslant 3/2$

$$f(x) = \begin{cases} -(x+2) + (3-2x) & x < -2 \\ (x+2) + (3-2x) & -2 \leq x < 3/2 \\ (x+2) - (3-2x) & x \geqslant 3/2 \end{cases}$$

$$f(x) = \begin{cases} -3x+1 \\ -x+5 \\ 3x-1 \end{cases}$$

$f(x) \leq x+5$

$\Rightarrow$

$-3x+1 \leq x+5 \qquad x < -2$

$-4 \leqslant 4x$

$x > -1 \qquad$ not possible

$-x+5 \leq x+5 \qquad -2 \leq x < 3/2$

$0 \leqslant 2x$

$x \geqslant 0$

$3x-1 < x+5 \qquad x \geqslant 3/2$

$2x \leq 6$

$x \leqslant 3 \qquad x \geqslant 3/2$

hence $\quad 0 \leq x \leq 3$

12

$f(x) = 2e^{0.25x}$

$f'(x) = 0.5e^{0.25x}$

$f'(8) = \frac{1}{2}e^{2}$

$f(8) = 2e^{2}$

$\frac{y - 2e^{2}}{x - 8} = \frac{1}{2}e^{2}$

$y - 2e^{2} = \frac{1}{2}e^{2}x - 4e^{2}$

$\Rightarrow \quad y = e^{2}\left[\frac{1}{2}x - 4 + 2\right]$

$= e^{2}\left[\frac{1}{2}x - 2\right]$

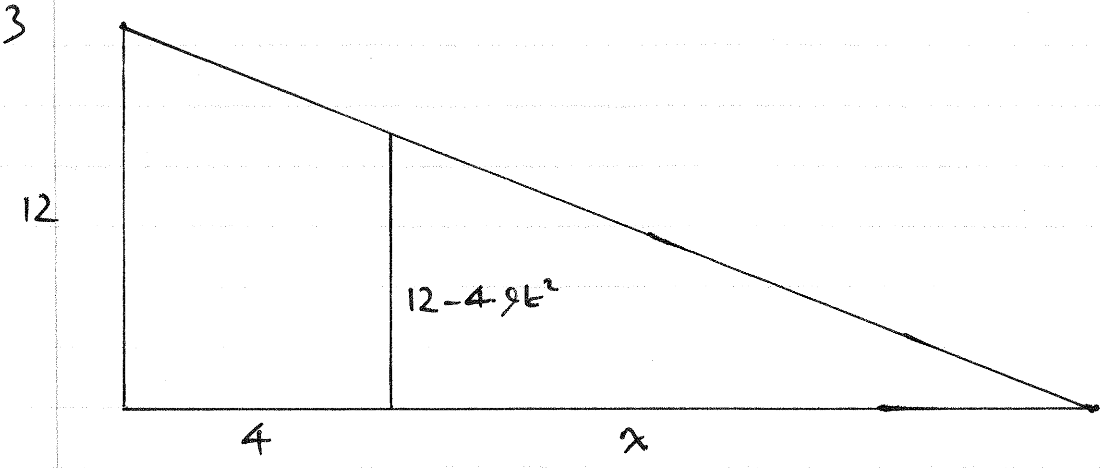
$A = \int_0^8 2e^{0.25x}\,dx - \int_4^8 e^{2}\left(\frac{1}{2}x - 2\right)dx$

$= 8e^{0.25x}\Big|_0^8 - \frac{1}{2} \times 4 \times 2e^{2}$

$= 8e^{2} - 8 - 4e^{2}$

$= 4e^{2} - 8$

13



$$\frac{x+4}{12} = \frac{x}{12-4.9t^2}$$

$$(x+4)(12-4.9t^2) = 12x$$

$$12x - 4.9t^2x + 48 - 19.6t^2 = 12x$$

$$-4.9t^2x + 48 - 19.6t^2 = 0$$

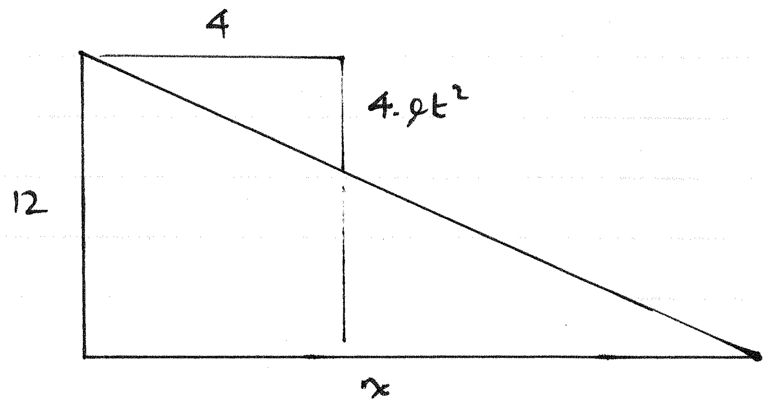
$$-9.8tx - 4.9t^2\frac{dx}{dt} - 39.2t = 0$$

$$\frac{dx}{dt} = \frac{39.2 + 9.8x}{4.9t}$$

$$t = 0.5, \quad x = 35.184$$

$$\left.\frac{dx}{dt}\right|_{t=0.5,\ x=35.184} = 156.74 \text{ m/sec}$$

13



$$\frac{x}{12} = \frac{4}{4.9t^2}$$

when $t = 0.5$
$x = 39.18$

$$4.9xt^2 = 48$$

$$4.9t^2\frac{dx}{dt} + 9.8tx = 0$$

$$\frac{dx}{dt} = -\frac{9.8tx}{4.9t^2}$$

$$\left.\frac{dx}{dt}\right|_{t=0.5,\ x=39.18} = 156.7$$

14

let $\sqrt{2} = \frac{a}{b}$ a, b integer no common factors (cannot both be even)

then $2 = \frac{a^2}{b^2}$

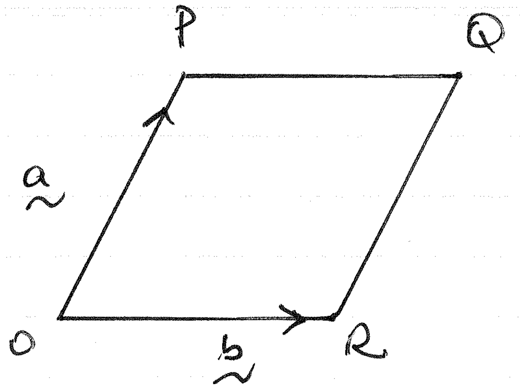
$\Rightarrow a^2 = 2b^2$ hence a^2 is even
therefore a is even

let $a = 2k$ since a is even
then $a^2 = 4k^2$

so $2 = \frac{4k^2}{b^2}$

$\Rightarrow b^2 = 2k^2$ hence b^2 is even
therefore b is even

14



let $\overrightarrow{OP} = \underset{\sim}{a}$

let $OR = \underset{\sim}{b}$

$$|\underset{\sim}{a}| = |\underset{\sim}{b}|$$

$$\overrightarrow{OQ} = \underset{\sim}{a} + \underset{\sim}{b}$$

$$\overrightarrow{PR} = \underset{\sim}{b} - \underset{\sim}{a}$$

$$\overrightarrow{OQ} \cdot \overrightarrow{PR} = (\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b}$$

$$= |b|^2 - |a|^2$$

$$= 0$$

hence diagonals are perpendicular

15

(a) $n = \overrightarrow{AE} = \overrightarrow{OE} - \overrightarrow{OA} = 12\hat{i} + 25j - 9k$

$= BF$

$= CG$

$= DH$

$c = \underset{\sim}{n} \cdot \overrightarrow{OE}$

$= (12\hat{i} + 25\hat{j} - 9k) \cdot (13\hat{i} + 27\hat{j} - 6k)$

$= 885$

$\underset{\sim}{r} \cdot (12\hat{i} + 25\hat{j} - 9k) = 885$

(b) $\underset{\sim}{r} = \overrightarrow{OA} + \lambda \overrightarrow{AE}$

$= \hat{i} + 2j + 3k + \lambda(12\hat{i} + 25j - 9k)$

$= (12\lambda + 1)\hat{i} + (25\lambda + 2)j + (-9\lambda + 3)k$

15 $\vec{AE} = \vec{OE} - \vec{OA}$

$= 12\hat{i} + 25\hat{j} - 9k$

$\vec{OP} = \vec{OA} + \lambda AE$

$= (12\lambda + 1)\hat{i} + (25\lambda + 2)\hat{j} + (-9\lambda + 3)k$

$\vec{DP} = OP - OD$

$= (12\lambda + 1 + 7)\hat{i} + (25\lambda + 2 - 8)\hat{j} + (-9\lambda + 3 - 9)k$

$= (12\lambda + 8)\hat{i} + (25\lambda - 6)\hat{j} + (-9\lambda - 6)k$

$OH = \vec{OD} + \vec{AE}$

$= 5\hat{i} + 33\hat{j} + 0k$

$HP = \vec{OP} - OH$

$= (12\lambda + 1 - 5)\hat{i} + (25\lambda + 2 - 33)\hat{j} + (-9\lambda + 3 + 0)k$

$= (12\lambda - 4)\hat{i} + (25\lambda - 31)\hat{j} + (-9\lambda + 3)k$

want $HP \cdot DP = 0$

$(12\lambda + 8)(12\lambda - 4) + (25\lambda - 6)(25\lambda - 31) + (-9\lambda - 6)(-9\lambda + 3) = 0$

$850\lambda^2 - 850\lambda + 136 = 0$

$\lambda = 0.2$ or 0.8

$\lambda = 0.2$, $\vec{OP} = 3.4\hat{i} + 7\hat{i} + 1.2k$

$AP = OP - OA$

$= (3.4 - 1)\hat{i} + (7 - 2)\hat{j} + (1.2 - 3)k$

$|AP| = \sqrt{34}$

$\lambda = 0.8$, $OP = 10.6\hat{i} + 22\hat{j} + -4.2k$

$AP = OP - OA$

$= (10.6 - 1)\hat{i} + (22 - 2)\hat{j} + (-4.2 - 3)k$

shortest distance is $\sqrt{34} = 5.831$

16 $\omega = \cos\theta + i\sin\theta$

$= e^{i\theta}$

$z = \cos\phi + i\sin\phi$

$= e^{i\phi}$

$\omega z = e^{i(\theta+\phi)}$

$= \cos(\theta+\phi) + i\sin(\theta+\phi)$

$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$= \cos\theta\cos\phi - \sin\theta\sin\phi + i[\sin\theta\cos\phi + \sin\phi\cos\theta]$

$\text{Im}(\omega z) = \sin(\theta+\phi)$

$= \sin\theta\cos\phi + \sin\phi\cos\theta$

hence $\sin(\theta+\phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$

$\int 3\cos(\theta+\pi/4)\left(\frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{2}}\right)^2 d\theta$

$= \int 3\cos(\theta+\pi/4)\left[\cos\theta\sin\frac{\pi}{4} + \sin\theta\cos\frac{\pi}{4}\right]^2 d\theta$

$= \int 3\cos(\theta+\pi/4)\left[\sin(\theta+\pi/4)\right]^2 d\theta$

$= \sin^3(\theta+\pi/4) + C$

17 $x = a\cos\frac{\pi t}{3}$

(a) $t = 1$ $\pm 12 = a\cos\frac{\pi}{3}$

$\Rightarrow \pm 12 = a\left(\frac{1}{2}\right)$

$\Rightarrow a = \pm 24$, but $a > 0$

$\Rightarrow a = 24$

(b) $x = 24\cos\frac{\pi t}{3}$

$\Rightarrow \dot{x} = \frac{\pi}{3}\left(-24\sin\frac{\pi t}{3}\right)$

$\Rightarrow \ddot{x} = \left(\frac{\pi}{3}\right)^2\left(-24\cos\frac{\pi t}{3}\right)$

$\Rightarrow \ddot{x} = -\left(\frac{\pi}{3}\right)^2 x$ $\Rightarrow$ motion is SHM.

(c) $x = 0$, $\cos\frac{\pi t}{3} = 0$

$\Rightarrow \frac{\pi t}{3} = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$

$\Rightarrow t = \frac{3}{2}, \frac{9}{4}, \ldots$

$t = 3/2$ $\dot{x} = \frac{\pi}{3}\left(-24\sin\left(\frac{\pi}{3}\cdot\frac{3}{2}\right)\right)$

$= -8\pi$

speed is 8π cm s^{-1}.

17 $$d = \int_0^{60} |\dot{x}|\, dt$$

$$= \int_0^{60} \left| -8\pi \sin \frac{\pi t}{3} \right| dt$$

$$= 960$$

or period $= \dfrac{2\pi}{\pi/3}$

$= 6$ seconds

hence 60 seconds is 10 cycles

since amplitude is 24, then 1 cycle covers 4×24

hence 10 cycles covers $4 \times 24 \times 10$

$= 960$

19 (a) $\frac{dx}{dt} = 3+4x \Rightarrow \int \frac{dx}{3+4x} = \int dt$

$\Rightarrow \frac{1}{4} \ln|3+4x| = t+c$ — ①

$t=1,\ x=2$

$\Rightarrow \frac{1}{4} \ln|11| = c+1$ — ⑪

eq① - eq⑪

$\frac{1}{4}\left[\ln|3+4x| - \ln 11\right] = t-1$

$\ln\left|\frac{3+4x}{11}\right| = 4t-4$

$3+4x = 11e^{4t-4}$

$4x = 11e^{4t-4} - 3$

$x = \frac{11}{4}e^{4t-4} - \frac{3}{4}$

(b) $x=3 \quad \frac{dx}{dt} = 15$

$t=3 \quad \frac{dx}{dt} = 11e^{4t-4}$

$\left.\frac{dx}{dt}\right|_{t=3} = 11e^{8}$

(c) $\frac{d^2x}{dt^2} = 4\frac{dx}{dt}$

$= 4[3+4x]$

$t=1,\ x=2$

$\frac{d^2x}{dt^2} = 44.$

CA

20 $p(n) = \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$

(a) $p(1) = 1$

$p(4) = 228$

(b) $p(1)$ is an integer

assume that $p(k)$ is an integer

when $n = k+1$

$$p(k+1) = \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$$

$$= \frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^3 + 3k^2 + 3k + 1}{3} + \frac{7k+7}{15}$$

$$= \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + \left[\frac{5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{3k^2 + 3k + 1}{3} + \frac{7}{15} \right]$$

$$= p(k) + \left[\frac{15k^4 + 30k^3 + 30k^2 + 15k + 3 + 15k^2 + 15k + 5 + 7}{15} \right]$$

$$= p(k) + \frac{1}{15}\left[15k^4 + 30k^3 + 45k^2 + 30k + 15\right]$$

$$= p(k) + \left(k^4 + 2k^3 + 3k^2 + 2k + 1\right)$$

since $p(k)$ is an integer and $k^4 + 2k^3 + 3k^2 + 2k + 1$ is always an integer when k is integer then $p(n)$ is always an integer

note

$$p(k+1) - p(k) = (k^2 + k + 1)^2$$

II

19 $$\frac{dx}{dt} = 3+4x$$

$$\frac{1}{4}\ln 11 = c+1$$

$$\Rightarrow \quad c = \frac{1}{4}\ln 11.$$

$$\frac{1}{4}\ln|3+4x| = t + \frac{1}{4}\ln 11$$

$$\ln|3+4x| = 4t + \ln 11$$

$$\Rightarrow \quad 3+4x = e^{4t+\ln 11}$$

$$= 11e^{4t}$$

$$x = \frac{1}{4}\left(11e^{4t} - 3\right)$$